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POTENTIAL-FLOW ANALYSIS OF UNSTEADY OUTFLOW FROM A TANK AND ITS EFFECT ON THE DYNAMICS OF A FLUID SYSTEM

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SUMMARY

When an analytical investigation of the unsteady flow from a propellant tank was made, a simple equation was obtained that relates the tank pressure drop to the development of the volume flow rate out of the tank. This equation allows the incorporation of the tank characteristics into a one-dimensional analysis of the dynamics of a fluid system and makes possible the determination of situations in which the dynamic characteristics of the tank are important.

Including the dynamic characteristics of the tank in a system analysis becomes more important as: (1) the ratio of liquid level to tank radius increases, (2) the ratio of outlet radius to tank radius increases, and (3) the ratio of outlet-pipe length to tank radius decreases. In traditional rocket systems the increase in accuracy obtained when the dynamic properties of the tank are included is usually only a few percent. In some systems, however, the dynamic effects of the tank can be important.

INTRODUCTION

In many fluid systems the tank pressure drop is a small fraction of the pressure drop throughout the system. Errors in the estimation of the tank pressure drop are often of little consequence. If, however, the temperature of the liquid in the tank is near boiling and if the liquid is flowing to a pump that is operating near its cavitation limit, then small errors in the estimation of tank pressure drop can make large differences in the estimation of pump performance. Furthermore, the starting-flow tank pressure drop for a tank connected by a short pipe to a receiver is a large part of the total system pressure drop. Under such conditions an improvement in the accuracy of tank pressure drop computations may be desired.

This report describes a potential-flow solution to determine the pressure drop during the starting flow out of a tank. This solution is limited to the case in which the free liquid surface in the tank is high enough for the surface to be considered flat and moving downward with the same speed at all points. A criterion is developed for determining when this condition on the free liquid surface exists. An illustrative example shows how the results of this report are used in the analysis of the starting flow in a fluid system involving tanks, lines, valves, and orifices.

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ANALYSIS

The analysis of the tank flow is restricted to a nonviscous fluid, so that real fluid effects (e.g., outlet separation) are not considered. It is assumed throughout the analysis that the fluid has no component of tangential velocity. In addition to the restrictions on the fluid, the analysis is restricted to tanks having flat bottoms.

Bernoulli Equation for Unsteady Flow

An unsteady, irrotational, incompressible flow satisfies the Bernoulli equation (ref. 1)

$$\frac{\partial}{\partial t}\varphi(x,y,z,t) + \frac{1}{2}q^2(x,y,z,t) + \frac{p(x,y,z,t)}{\rho} + yg = f(t) \qquad (1)$$

where q is the magnitude of the local velocity vector $\bar{q}$; φ is the velocity potential, that is, gradient $\varphi(x,y,z,t) = \bar{q}(x,y,z,t)$; and $f(t)$ is a function of time. (All symbols are defined in the appendix and the coordinate axes are shown in fig. 1.) Since the pressure level in the entire flow field can be varied with time in any way without otherwise affecting the flow, $f(t)$ is arbitrary.

A cylindrical tank oriented as shown in figure 1 is analyzed. Two locations on the tank centerline or y-axis, that may vary with time are denoted by $y_1(t)$ and $y_2(t)$.

Equation (1) can be evaluated at the two locations at the same instant and the function $f(t)$ eliminated; the result is

$$\frac{\partial}{\partial t}\left\{\varphi[y_1(t),t] - \varphi[y_2(t),t]\right\} + \frac{v^2[y_1(t),t] - v^2[y_2(t),t]}{2}$$
$$+ \frac{p[y_1(t),t] - p[y_2(t),t]}{\rho} + gy_1(t) - gy_2(t) = 0 \qquad (2)$$

where v is the vertical component of velocity.

For this analysis $y_2(t)$ is taken to be the origin of coordinates (fig. 1), which corresponds to the center of the outlet pipe at the bottom of the tank; therefore, $y_2(t)$ is invariant with time. Equation (2), then, becomes

$$\frac{\partial}{\partial t}\left\{\varphi[y_1(t),t] - \varphi(0,t)\right\} + \frac{v^2[y_1(t),t] - v^2(0,t)}{2} + \frac{p[y_1(t),t] - p(0,t)}{\rho} + gy_1(t) = 0 \qquad (3)$$

Method of Determining Tank Potential Function

In order to proceed with the solution of equation (3), some knowledge of the potential function along the y-axis is required. In what follows $\varphi(y,t)$ will be determined through an approximation of the desired tank flow using an infinite number of sinks.

An infinite set of three-dimensional point sinks distributed in the x,z-plane (i.e., the plane of the tank bottom) in the pattern shown in figure 2(a) repeated indefinitely is considered. The plane passing through points A and C perpendicular to the x,z-plane is a plane of symmetry in the resulting flow. Since there can be no flow through this plane without destroying the symmetry, the plane is a stream surface. This is also true for similar planes through points B and D, points C and E, and so forth. It follows that a hexagonal tank erected perpendicular to the x,z-plane on the base, which is outlined by a solid line in figure 2(a), is actually a stream surface of the flow. Since the x,z-plane is also a stream surface, the flow can be interpreted as the flow out of a point in the center of the bottom of a hexagonal tank of infinite length and finite diameter $(2r_T')$. The flow field induced by the sinks is actually a honeycomb of such hexagonal tanks, but attention is confined to the one that contains the origin of the coordinates (see fig. 1). (It is interesting to note that there are many other planes of symmetry and that, if the proper ones are selected for tank walls, various other tank configurations can be obtained from this same flow field. In particular, a hexagonal tank can be formed with outlets at three corners instead of one outlet at the center. The results obtained in this paper can, therefore, be applied to such a case with the obvious difference that three times the volume flow is leaving this new tank.)

For the purpose of this investigation the point-sink approximation (fig. 2(a)) to the flow from the tank (fig. 1) is deficient in two ways. First, the tank is hexagonal instead of cylindrical. Since the tank walls are a substantial distance from the outlet in most practical tanks, however, this disadvantage is not considered too important. The second and more serious drawback is the representation of the tank outlet by a point sink, which results in infinitely large velocities at the outlet and gives no indication of the possible effect of outlet radius. To improve this situation the point sink inside the tank can be replaced by a uniformly distributed sink that covers the actual area of the tank outlet πr_{out}^2 (fig. 2(b)). Within the outlet area in the x,z-plane, the uniformly distributed sink will have a constant finite vertical velocity equal to the volume flow rate of the replaced point sink divided by the area of the tank outlet.

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It will be shown that the differences between a point sink and a distributed sink diminish very rapidly beyond the immediate neighborhood of the sink. With this justification a great simplification is achieved by the continued use of point sinks outside the tank. This results in a lack of complete symmetry about the planes forming the tank walls (see fig. 2(b)), so that the walls are distorted to some degree, especially near the base. This distortion is expected to be negligible for tank- to outlet-radius ratios of practical interest.

One further point needs discussion. The sink-induced flow, which is actually the flow from an infinitely long tank of liquid, is used to approximate the flow from a tank having a finite liquid level. The essential features of the free-surface flow (i.e., the flow with a finite liquid level) are the following:

(1) The pressure is constant on the free surface.

(2) The free surface is always composed of the same particles.

In general, the particles that form a constant-pressure surface at one instant of time in the sink-induced flow will not form a constant-pressure surface at other times. The sink-induced flow therefore deviates from the flow that would have resulted if the free-surface conditions had been included in the analysis; however, the velocity of the sink-induced flow is uniform across the tank at a location sufficiently above the bottom of the tank. In this case, a surface that is normal to the tank axis will be a constant-pressure surface and will continue to be one as long as the velocity remains uniform. The investigation is therefore confined to sufficiently great liquid levels (greater than 1 tank diam), so that the sink-induced flow is a suitable substitute for the actual free-surface flow.

Determination of Potential Function

The potential function for a single point sink located at an arbitrary point (x_o,y_o,z_o) and having a volume flow rate of $Q(t)$ cubic feet per second coming from the half-space above the x,z-plane is:

$$\varphi(x,y,z,t) = \frac{Q(t)}{2\pi\sqrt{(x - x_o)^2 + (y - y_o)^2 + (z - z_o)^2}} + \text{constant} \tag{4}$$

(When the first 2 in the denominator is replaced by a 4, the customary equation for a point sink of strength $Q(t)$ in the whole space results.) If only the values of φ on the y-axis and only sinks located in the x,z-plane are considered, equation (4) becomes

$$\varphi(y,t) = \frac{Q(t)}{2\pi\sqrt{x_o^2 + y^2 + z_o^2}} + \text{constant} \tag{5}$$

If φ_Σ denotes the contribution of all point sinks outside the tank, it follows from figure 2 that

$$\varphi_\Sigma(y,t) = \frac{Q(t)}{2\pi r_T'}\left[\sum_{k=1}^{\infty}\frac{2}{\sqrt{\left(\frac{y}{r_T'}\right)^2 + 4k^2}} + \sum_{k=1}^{\infty}\frac{2}{\sqrt{\left(\frac{y}{r_T'}\right)^2 + 12k^2}}\right.$$

$$+ \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{4}{\sqrt{\left(\frac{y}{r_T'}\right)^2 + 3(2k - 1)^2 + (2j - 1)^2}}$$

$$\left.+ \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{4}{\sqrt{\left(\frac{y}{r_T'}\right)^2 + 12k^2 + 4j^2}}\right] + \text{constant} \tag{6}$$

where the terms in their respective order represent the sinks on the z-axis, the sinks on the x-axis, the sinks on the odd-numbered rows, and the sinks on the even-numbered rows. In its present form the series (eq. (6)) diverges for all values of y/r_T'. If, however, an appropriate constant is associated with each individual sink, the result is the following convergent series:

$$\varphi_\Sigma(y,t) = \frac{Q(t)}{2\pi r_T'}\left\{\sum_{k=1}^{\infty}\left[\frac{2}{\sqrt{\left(\frac{y}{r_T'}\right)^2 + 4k^2}} - \frac{2}{\sqrt{4k^2}}\right] + \sum_{k=1}^{\infty}\left[\frac{2}{\sqrt{\left(\frac{y}{r_T'}\right)^2 + 12k^2}} - \frac{2}{\sqrt{12k^2}}\right]\right.$$

$$+ \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left[\frac{4}{\sqrt{\left(\frac{y}{r_T'}\right)^2 + 3(2k-1)^2 + (2j-1)^2}} - \frac{4}{\sqrt{3(2k-1)^2 + (2j-1)^2}}\right]$$

$$\left. + \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left[\frac{4}{\sqrt{\left(\frac{y}{r_T'}\right)^2 + 12k^2 + 4j^2}} - \frac{4}{\sqrt{12k^2 + 4j^2}}\right]\right\} \tag{7}$$

The distributed sink about the origin is obtained in the following way. If r_{out} denotes the radius of the outlet pipe, the volume flow rate per unit area in the outlet is $Q/\pi r_{out}^2$. If $s = \sqrt{x^2 + z^2}$ and $\theta = \tan^{-1}\frac{x}{z}$ are polar coordinates in the plane of the outlet, the volume flow rate through an increment of area $s\,\Delta\theta\,\Delta s$ is $(Q/\pi r_{out}^2)s\,\Delta\theta\,\Delta s$. According to equation (5), the contribution to the potential function on the y-axis, which arises from the increment $s\,\Delta\theta\,\Delta s$ of area in the outlet region, is approximately

$$\frac{\frac{Q}{\pi r_{out}^2}\, s\,\Delta\theta\,\Delta s}{2\pi\sqrt{s^2 + y^2}}$$

where the sink is assumed to be located at a single point within the increment of area. The potential function, arising from sinks uniformly distributed over the entire outlet area, is obtained when the various contributions (similar to the one just mentioned) are summed and Δs and $\Delta\theta$ approach zero. This potential is denoted by φ_c; the result is the integral

$$\varphi_c(y,t) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{r_{out}} \frac{\frac{Q(t)}{\pi r_{out}^2}\, s\, ds\, d\theta}{\sqrt{s^2 + y^2}} + \text{constant} \tag{8}$$

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When equation (8) is integrated and when the arbitrary constant is disregarded,

$$\varphi_c(y,t) = \frac{Q(t)}{\pi r_{out}^2}\left(\sqrt{r_{out}^2 + y^2} - y\right) = \frac{Q(t)}{\pi r_T'}\left(\frac{r_T'}{r_{out}}\right)^2\left[\sqrt{\left(\frac{r_{out}}{r_T'}\right)^2 + \left(\frac{y}{r_T'}\right)^2} - \frac{y}{r_T'}\right] \tag{9}$$

For $y > r_{out}$ equation (9) can be expanded in terms of r_{out}^2/y^2 as

$$\varphi_c(y,t) = \frac{Q(t)}{2\pi y}\left(1 - \frac{1}{4}\frac{r_{out}^2}{y^2} + \frac{1}{8}\frac{r_{out}^4}{y^4} \mp \ldots\right) \tag{10}$$

The first term on the right side is the equivalent of a point sink at the origin (see eq. (5)). The difference in the effects of a point sink and a distributed sink on the potential function is less than $r_{out}^2/4y^2$, or approximately 1 percent at 5 tank-outlet radii above the center of the distributed sink. This result tends to justify the use of point sinks instead of distributed sinks outside the tank.

From equations (7) and (9) it is seen that $\varphi r_T'/Q$, where

$$\frac{\varphi r_T'}{Q} \equiv \frac{\varphi_c r_T'}{Q} + \frac{\varphi_\Sigma r_T'}{Q},$$

depends only on y/r_T' and r_{out}/r_T'. In fact, r_{out}/r_T' only affects the value of $\varphi_c r_T'/Q$.

The velocities induced along the y-axis by the external sinks and by the central sink are obtained by differentiating equations (7) and (9), respectively:

$$\begin{aligned}\frac{r_T'^2 v_\Sigma(y,t)}{Q(t)} = \frac{-1}{\pi}\left(\frac{y}{r_T'}\right)\Bigg\{&\sum_{k=1}^{\infty}\left[\left(\frac{y}{r_T'}\right)^2 + 4k^2\right]^{-3/2} + \sum_{k=1}^{\infty}\left[\left(\frac{y}{r_T'}\right)^2 + 12k^2\right]^{-3/2} \\ &+ \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} 2\left[\left(\frac{y}{r_T'}\right)^2 + 3(2k-1)^2 + (2j-1)^2\right]^{-3/2} \\ &+ \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} 2\left[\left(\frac{y}{r_T'}\right)^2 + 12k^2 + 4j^2\right]^{-3/2}\Bigg\}\end{aligned} \tag{11}$$

$$\frac{r_T'^2 v_c(y,t)}{Q(t)} = \frac{1}{\pi}\left(\frac{r_T'}{r_{out}}\right)^2 \left[\frac{y}{r_T'}\left(\frac{r_{out}^2}{r_T'^2} + \frac{y^2}{r_T'^2}\right)^{-1/2} - 1\right] \tag{12}$$

Computations. - Since the series (eq. (7)) converges slowly, it is not very useful for computations. In order to obtain values of φ_Σ, the velocity v_Σ was first computed by means of equation (11). With v_Σ as a function of y, it was possible to integrate and get φ_Σ. Equation (7) was used to spot check the values of φ_Σ obtained by integration.

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Figure 3 shows the calculated variations of $r_T'\varphi_\Sigma/Q$ and $r_T'\varphi_c/Q$ with y/r_T' for two extreme values of r_{out}/r_T'. For all values of y/r_T' greater than 2.0 (1 tank diam), the effect of r_{out}/r_T' disappears. The effect of φ_c on the variation of the total potential function $\varphi_c + \varphi_\Sigma$ is large near the outlet but rapidly diminishes when y/r_T' increases. The effect of φ_Σ on the variation of $\varphi_c + \varphi_\Sigma$ is small near the outlet, but φ_Σ quickly becomes the principal element. The sum of these two potentials is nearly linear from a value of y/r_T' of approximately 1.5.

Figure 4 shows the variation of the velocity ratio v/V along the tank axis, where $V = -Q/2\sqrt{3}\, r_T'^2$ is the magnitude of the velocity far up in the tank. (The minus sign occurs because the flow direction is opposite to the positive y-axis.) Two extreme values of r_{out}/r_T' were used in equations (11) and (12) to obtain the plot. The entire velocity is shown in figure 4(a), and the contributions of the external sinks and of the central sink are shown separately in figure 4(b). The figure shows that the velocity at the centerline of the tank quickly approaches the uniform-flow velocity with increasing height above the outlet. In fact, v (the total velocity in fig. 4(a)) is within 1 percent of V by $y/r_T' = 1.5$. This indicates that essentially uniform flow is occurring less than 1 tank diameter up from the bottom, if it is assumed that maximum deviation occurs at the center of the tank. This result is not affected by the value of r_{out}/r_T'. The use of the sink flow for tank liquid levels of 1 diameter or more is justified.

Potential-function difference $\varphi(y,t) - \varphi(0,t)$. - Consider only those values of y so far above the tank outlet that the flow is uniform and has the velocity $v = V$. Since $v = \frac{\partial\varphi}{\partial y}$, φ is independent of x and z and varies linearly with y for all such large values of y.

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Therefore,

$$\varphi(y,t) \equiv \varphi_c(y,t) + \varphi_\Sigma(y,t) = V_y + A_o \tag{13}$$

where A_o is independent of y. Equation (13) is valid only for those values of y so large that the flow at that level can be considered uniform. The value of A_o can be determined from previously computed values of $\varphi_c + \varphi_\Sigma$, which are presented in figure 3, as follows. Since $Q = -2\sqrt{3}\, r_T'^2 V$, equation (13) can be rewritten as

$$\frac{r_T'(\varphi_c + \varphi_\Sigma)}{Q} = \frac{-1}{2\sqrt{3}} \frac{y}{r_T'} + \frac{r_T' A_o}{Q} \tag{14}$$

When the value of $r_T' A_o/Q$ is determined from the values given in figure 3 in such a way that equation (14) fits the straight portion of the curve, equation (14) becomes

$$\varphi(y,t) = \varphi_c(y,t) + \varphi_\Sigma(y,t) = \frac{Q(t)}{r_T'}\left(0.344 - \frac{1}{2\sqrt{3}} \frac{y}{r_T'}\right) \tag{15}$$

Equation (15) is a valid expression for $\varphi(y,t)$ as long as y is sufficiently large.

Since $\varphi_\Sigma(0,t) = 0$, the value of $\varphi(y,t)$ at $y = 0$ derived from equation (9) is

$$\varphi(0,t) = \varphi_c(0,t) = \frac{Q(t)}{r_T'\pi}\left(\frac{r_T'}{r_{out}}\right) \tag{16}$$

When equations (15) and (16) are combined, the result is

$$\varphi(y,t) - \varphi(0,t) = \frac{Q(t)}{r_T'}\left(0.344 - \frac{1}{2\sqrt{3}} \frac{y}{r_T'} - \frac{1}{\pi} \frac{r_T'}{r_{out}}\right) \tag{17}$$

which is valid for all y sufficiently large.

Simplified Bernoulli Equation for Unsteady Tank Flow

As long as $y_1(t)$ is sufficiently large, equation (17) can be used in equation (3) to give

$$\frac{1}{r_T'}\left(0.344 - \frac{1}{2\sqrt{3}}\frac{y_1}{r_T'} - \frac{1}{\pi}\frac{r_T'}{r_{out}}\right)\frac{dQ(t)}{dt} - \frac{1}{24r_T'^4}\left(\frac{12}{\pi^2}\frac{r_T'^4}{r_{out}^4} - 1\right)Q^2(t)$$

$$+ \frac{p(y_1) - p(0)}{\rho} + gy_1 = 0 \qquad (18)$$

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where

$$Q = -2\sqrt{3}\ r_T'^2\ v(y_1) = -\pi r_{out}^2 v(0)$$

and

$$\frac{v^2(y_1) - v^2(0)}{2} = \frac{1}{2}\left[\left(\frac{Q(t)}{2\sqrt{3}\ r_T'^2}\right)^2 - \left(\frac{Q(t)}{\pi r_{out}^2}\right)^2\right] = \frac{Q^2(t)}{24r_T'^4}\left(1 - \frac{12}{\pi^2}\frac{r_T'^4}{r_{out}^4}\right)$$

$$(19)$$

Since r_T' is a characteristic dimension of a hexagonal tank, it is convenient to replace r_T' with an equivalent radius r_T (i.e., the radius of a cylindrical tank of equal cross-sectional area). Since $2\sqrt{3}\ r_T'^2 = \pi r_T^2$, equation (18) becomes

$$\frac{1}{r_T}\left(0.361 - \frac{1}{\pi}\frac{r_T}{r_{out}} - \frac{1}{\pi}\frac{y_1}{r_T}\right)\frac{dQ(t)}{dt} - \frac{1}{2\pi^2 r_T^4}\left(\frac{r_T^4}{r_{out}^4} - 1\right)Q^2(t)$$

$$+ \frac{p(y_1) - p(0)}{\rho} + gy_1 = 0 \qquad (20)$$

Equation (20) is the principal equation of this analysis and includes, with some simplifications, the effect of the tank dynamics in the expression for the tank pressure drop. The form of the equation is typical of a one-dimensional system analysis and is easily incorporated into such an analysis. It should be remembered that equation (20) applies only for values of y_1 of approximately 1 tank diameter or more.

To use this equation in tank-flow analyses, $y_1(t)$ is taken equal to $Y(t)$, which is the height of the free surface (i.e., the depth of the liquid in the tank). In this case equation (20) has two unknowns, $Y(t)$ and $Q(t)$, which are related by means of the differential equation.

$$\frac{dY(t)}{dt} = V(t) = \frac{Q(t)}{\pi r_T^2} \qquad (21)$$

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In general, equations (20) and (21) can be solved simultaneously when $p(Y) - p(0)$ is given or determined by one or more additional differential equations. For starting transients of very short duration, however, the change in Y is negligible and equation (20) can be used alone.

Dynamics of Tank and Line

If the tank flow is considered steady as in a simplified system analysis, Bernoulli's equation can be used to obtain

$$\frac{\Delta p_T}{\rho} = \frac{Q^2(t)}{2\pi^2 r_T^4}\left(\frac{r_T^4}{r_{out}^4} - 1\right) - gY \tag{22}$$

The increased accuracy of equation (20) over equation (22) is entirely in the coefficient of $dQ(t)/dt$.

Now if a pipe of radius r_{out} and length L is attached to the outlet, the pressure drop along the pipe is

$$\frac{\Delta p_P}{\rho} = \frac{L}{\pi r_{out}^2}\frac{dQ(t)}{dt} + \frac{Lf}{4\pi^2 r_{out}^5} Q^2(t) \tag{23}$$

The total drop in pressure across the tank and across the length of pipe L is

$$\frac{\Delta p_{TOT}}{\rho} = \frac{1}{r_T}\left(\frac{r_T L}{\pi r_{out}^2}\right)\frac{dQ(t)}{dt} + \frac{Lf}{4\pi^2 r_{out}^5} Q^2(t) + \frac{1}{2\pi^2 r_T^4}\left(\frac{r_T^4}{r_{out}^4} - 1\right) Q^2(t) - gY \tag{24}$$

If equation (20) is used instead of the steady-flow Bernoulli equation (eq. (22)), the expression for pressure drop becomes

$$\frac{\Delta p_{TOT}}{\rho} = \frac{1}{r_T}\left(\frac{r_T L}{\pi r_{out}^2} + \frac{1}{\pi}\frac{r_T}{r_{out}} + \frac{1}{\pi}\frac{Y}{r_T} - 0.361\right)\frac{dQ(t)}{dt} + \frac{Lf}{4\pi^2 r_{out}^5} Q^2(t)$$

$$+ \frac{1}{2\pi^2 r_T^4}\left(\frac{r_T^4}{r_{out}^4} - 1\right) Q^2(t) - gY \tag{25}$$

Equations (24) and (25) differ only in the coefficient of $dQ(t)/dt$. If K_{TOT} denotes this coefficient in equation (25),

$$K_{TOT} \equiv \frac{1}{r_T}\left(\frac{r_T L}{\pi r_{out}^2} + \frac{1}{\pi}\frac{r_T}{r_{out}} + \frac{1}{\pi}\frac{Y}{r_T} - 0.361\right) = \frac{1}{\pi r_T}\left(\frac{r_T}{r_{out}}\frac{L}{r_{out}} + \frac{r_T}{r_{out}} + \frac{Y}{r_T} - 1.134\right)$$

If the part of the coefficient that is attributable to the tank is denoted by K_T,

$$\frac{K_T}{K_{TOT}} = \frac{\frac{r_T}{r_{out}} + \frac{Y}{r_T} - 1.134}{\left(\frac{r_T}{r_{out}}\right)^2 \frac{L}{r_T} + \frac{r_T}{r_{out}} + \frac{Y}{r_T} - 1.134} \tag{26}$$

This ratio is a measure of the effect of the tank on the system dynamics and depends on the three ratios r_{out}/r_T, Y/r_T, and L/r_T.

Figure 5 is a plot of this ratio of coefficients. For any values of liquid level ratio Y/r_T and outlet-radius ratio r_{out}/r_T, the effect of including the tank dynamics falls off quickly as the outlet-pipe-length ratio L/r_T increases. The effect of the tank is greatest for the largest values of Y/r_T and r_{out}/r_T. For most cases of interest it appears from figure 5 that the effect of the tank is small. The tank begins to make a significant difference in the coefficient of dQ/dt for either very short outlet pipes or large liquid levels.

SAMPLE CALCULATION

In order to examine the effect of the unsteady flow term in a specific case, the following example is considered. Figure 6 is a schematic diagram of a part of a simple rocket system. The tank containing one of the propellants, the line that leads from the tank to the combustion chamber, the valve, and the injector are the only parts involved.

The pressure change from the ullage space in the tank (station 1) to the tank outlet (station 2) is given by equation (20) with $y_1 = Y$:

$$\frac{p_1 - p_2}{\rho g} = \frac{1}{r_T g}\left(\frac{1}{\pi}\frac{r_T}{r_{out}} + \frac{1}{\pi}\frac{Y}{r_T} - 0.361\right)\frac{dQ(t)}{dt} + \frac{1}{2\pi^2 r_T^4 g}\left(\frac{r_T^4}{r_{out}^4} - 1\right)Q^2(t) - Y \tag{27}$$

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The change in pressure from the tank outlet to the valve inlet (station 3) is given by (ref. 2)

$$\frac{p_2 - p_3}{\rho g} = \frac{L}{\pi r_{out}^2 g} \frac{dQ(t)}{dt} + \frac{Lf}{4\pi^2 r_{out}^5 g} Q^2(t) \tag{28}$$

The change in head across the valve is given by reference 2 as

$$\frac{p_3 - p_4}{\rho g} = \mathcal{R}_V Q^2(t) \tag{29}$$

In this example, valve resistance varies from that for a near off position, which just supports combustion, to that for a full on position, which admits the design volume flow rate. The resistance of the valve is a function of the valve design and varies with time in a prescribed way.

The change in pressure across the injector is given by

$$\frac{p_4 - p_5}{\rho g} = \mathcal{R}_{inj} Q^2(t) \tag{30}$$

where $\mathcal{R}_{inj}$ is injector resistance, a constant depending only on the particular design.

The temperature in the rocket chamber is assumed constant, and the throat is assumed to be choked. In this case the chamber head is directly proportional to the volume flow rate (ref. 2)

$$\frac{p_5}{\rho g} = R_{ch} Q(t) \tag{31}$$

If equations (29) to (31) are added together, the result is

$$\begin{aligned}\frac{p_1}{\rho g} &= \frac{1}{r_T g}\left(\frac{1}{\pi}\frac{y}{r_T} + \frac{1}{\pi}\frac{r_T}{r_{out}} + \frac{1}{\pi}\frac{r_T L}{r_{out}^2} - 0.361\right)\frac{dQ(t)}{dt} \\ &+ \left[\frac{1}{2\pi^2 r_T^4 g}\left(\frac{r_T^4}{r_{out}^4} - 1\right) + \frac{Lf}{4\pi^2 r_{out}^5 g} + \mathcal{R}_V + \mathcal{R}_{inj}\right] Q^2(t) + R_{ch} Q(t) - Y\end{aligned} \tag{32}$$

As liquid flows from the tank, the surface level Y changes; therefore, equation (32) must be used in conjunction with equation (21) to obtain a solution. A solution is first obtained, however, when this change in Y is ignored.

If Y is a constant, equation (32) can be solved numerically when the tank pressure p_1 and the valve resistance $\mathcal{R}_V$ are prescribed as functions of time. If p_1 is taken to be constant and the change in $\mathcal{R}_V$ is taken to be a step change, equation (32) can be solved in a closed form.

The following values are assumed and are consistent with a 15,000-pound-thrust rocket:

Tank pressure, p_1, lb/sq in.	500
Design chamber pressure, $p_5(t)$ as $t \to \infty$, lb/sq in.	338
Design flow rate, $Q(t)$ as $t \to \infty$, cu ft/sec	0.534
Starting flow rate, cu ft/sec	0.053
Injector resistance, $\mathcal{R}_{inj}$, sec^2/ft^5	578
Tank radius, r_T, ft	1.5
Liquid level, Y, ft	9.0
Outlet radius, r_{out}, ft	0.15
Constant of proportionality for rocket chamber, R_{ch}, sec/sq ft . .	1288
Pipe length, L, ft	1.5
Gravitational constant, g, ft/sec^2	32.2
Friction factor, f, dimensionless	0.025
Valve resistance in open position, $\mathcal{R}_V$, sec^2/ft^5	606
Propellant density, ρ, slugs/cu ft	2.2

When these values are substituted in equation (32), the result is

$$\frac{dQ}{dt} = 1354 - 1701Q - 1568Q^2 \tag{33}$$

With $Q = 0.053$ cubic foot per second at $t = 0$, equation (33) has the solution

$$Q = 1.076 \tanh (1687t + 0.623) - 0.542 \tag{34}$$

If the effect of the tank dynamics is neglected, the solution is

$$Q = 1.076 \tanh (1938t + 0.623) - 0.542 \tag{35}$$

Both of these results (eqs. (34) and (35)) are plotted in figure 7. A maximum difference of about 5 percent occurs in the volume flow rates for the two equations. Larger values of outlet radius will increase this difference, and larger values of pipe length will diminish it.

In this problem the time interval of interest is about 0.002 second (see fig. 7). During this time, the change in Y is less than 0.00015 foot. Since such a change would have a negligible influence on the simultaneous numerical solution of equations (32) and (21), the treatment of Y as a constant is justified. In many practical cases, Y is much smaller than $p_1/\rho g$ in equation (32), so that the variation of Y can hardly be a factor of importance. In other cases, however, $p_1/\rho g$ is small, and the variation of Y with time may not be negligible.

CONCLUSIONS

The following conclusions were drawn from an analytical investigation of the unsteady flow from a propellant tank:

1. A method is presented for finding the unsteady potential flow from a tank. The result is used to develop an equation relating the pressure drop (from the free surface of the liquid to the tank outlet) to the rate of change of volume flow. This equation allows the inclusion of tank dynamic effects in a system analysis, or the determination of the error involved in the treatment of the tank flow as steady.

2. The importance of including the tank dynamics in a system analysis increases as:

(a) The ratio of liquid level to tank radius increases

(b) The ratio of outlet radius to tank radius increases

(c) The ratio of outlet-pipe length to tank radius decreases

Lewis Research Center
National Aeronautics and Space Administration
Cleveland, Ohio, April 26, 1962

APPENDIX - SYMBOLS

A_o constant term in linear equation for $\varphi(y,t)$ at large y, sq ft/sec

f dimensionless friction factor

$f(t)$ arbitrary function of time

g gravitational constant, ft/sec^2

j summation index

K_{TOT} coefficient of dQ/dt in expression for pressure drop across a tank and line, ft^{-3}

K_T coefficient of dQ/dt in expression for tank pressure drop, ft^{-3}

k summation index

L length of outlet line, ft

p pressure, lb/sq ft

p_1 pressure at free surface of liquid in tank, lb/sq ft

p_2 pressure at center of tank bottom, lb/sq ft

p_3 pressure upstream of valve, lb/sq ft

p_4 pressure downstream of valve, lb/sq ft

p_5 pressure in rocket chamber, lb/sq ft

Δp_p pressure drop along tank-outlet line, lb/sq ft

Δp_T pressure drop from tank free surface to tank outlet, lb/sq ft

Δp_{TOT} pressure drop across tank and outlet line, lb/sq ft

Q volume flow rate, cu ft/sec

q magnitude of velocity vector, ft/sec

$\bar{q}$ local velocity vector, ft/sec

R_{ch} constant of proportionality for rocket chamber, sec/sq ft

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$\mathcal{R}_{inj}$	resistance of injector, sec^2/ft^5
$\mathcal{R}_V$	resistance of valve, sec^2/sq ft
r_{out}	radius of tank-outlet line, ft
r_T	radius of tank, ft
r'_T	one-half the width of hexagonal tank, ft
s	polar coordinate, radians
t	time, sec
V	velocity far up in tank, ft/sec
v	vertical component of velocity, ft/sec
v_c	vertical component of velocity induced by central distributed sink, ft/sec
v_Σ	vertical component of velocity induced by infinite set of point sinks, ft/sec
x	coordinate axis (fig. 1), ft
x_o	x-coordinate of typical point sink, ft
Y	y-coordinate of free surface, ft
y	coordinate axis (fig. 1), ft
y_o	y-coordinate of typical point sink, ft
y_1	arbitrary location on y-axis, ft
y_2	arbitrary location on y-axis, ft
z	coordinate axis (fig. 1), ft
z_o	z-coordinate of typical point sink, ft
θ	polar coordinate, radians
ρ	density, slugs/cu ft
φ	velocity potential, sq ft/sec

φ_c velocity potential of central distributed sink, sq ft/sec

φ_Σ velocity potential of infinite set of point sinks, sq ft/sec

REFERENCES

1. Stoker, J. J.: Water Waves. Interscience Publ., Inc., 1957.

2. Krebs, Richard P.: Effect of Fluid-System Parameters on Starting Flow in a Liquid Rocket. TN 4034, 1957.

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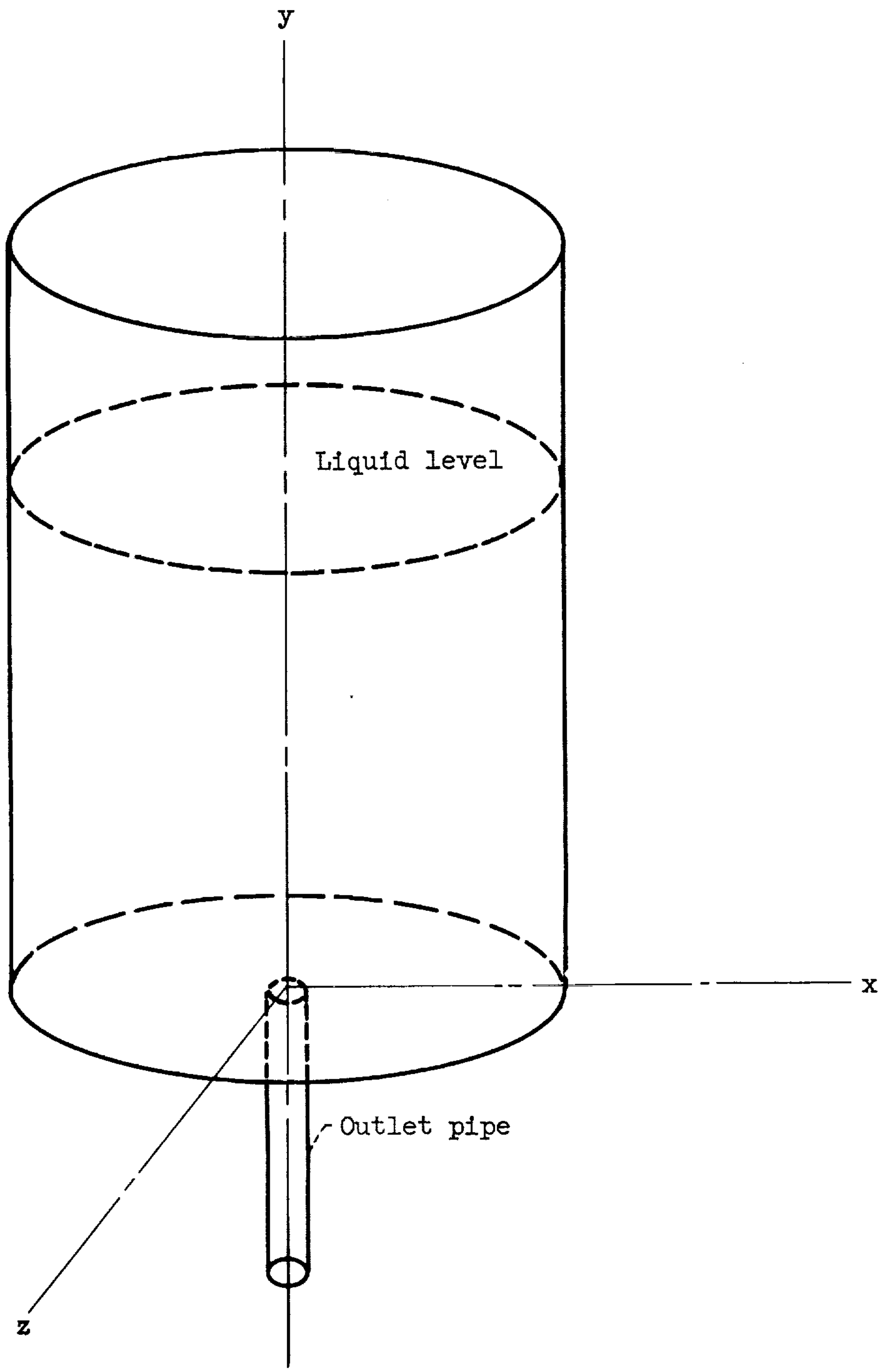

Figure 1. - Schematic of cylindrical tank with centrally located outlet pipe and coordinate system.

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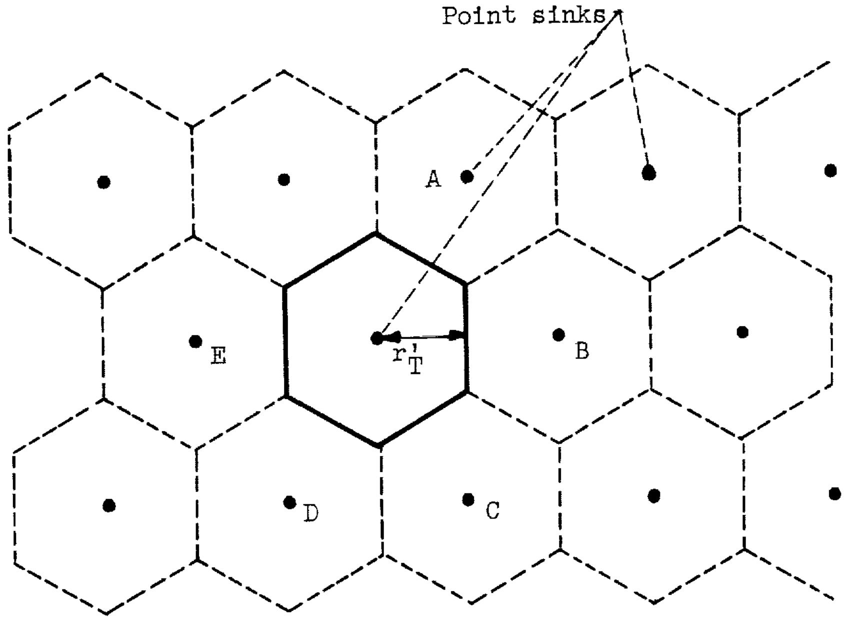
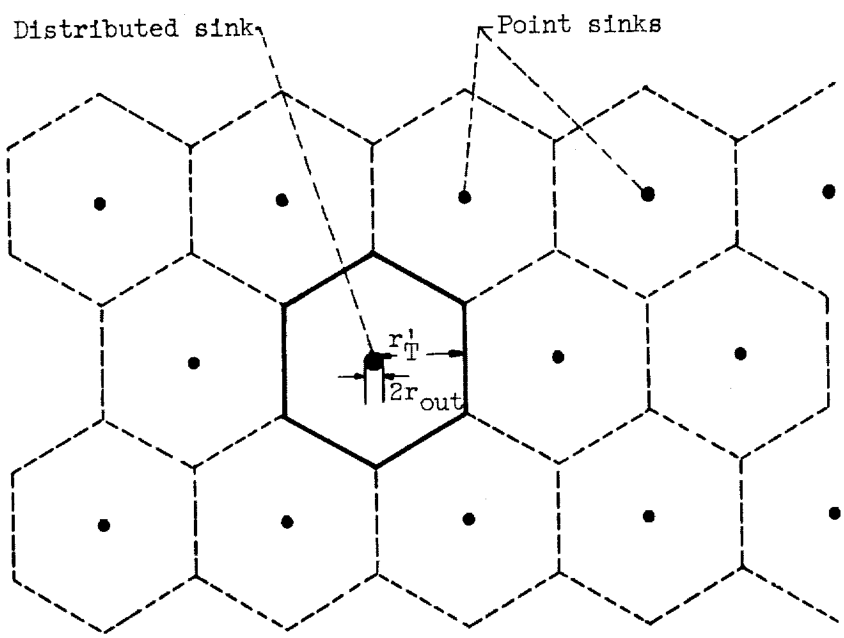


(a) Central point sink.



(b) Central distributed sink.

Figure 2. - Pattern of sinks used to obtain flow from hexagonal tank.

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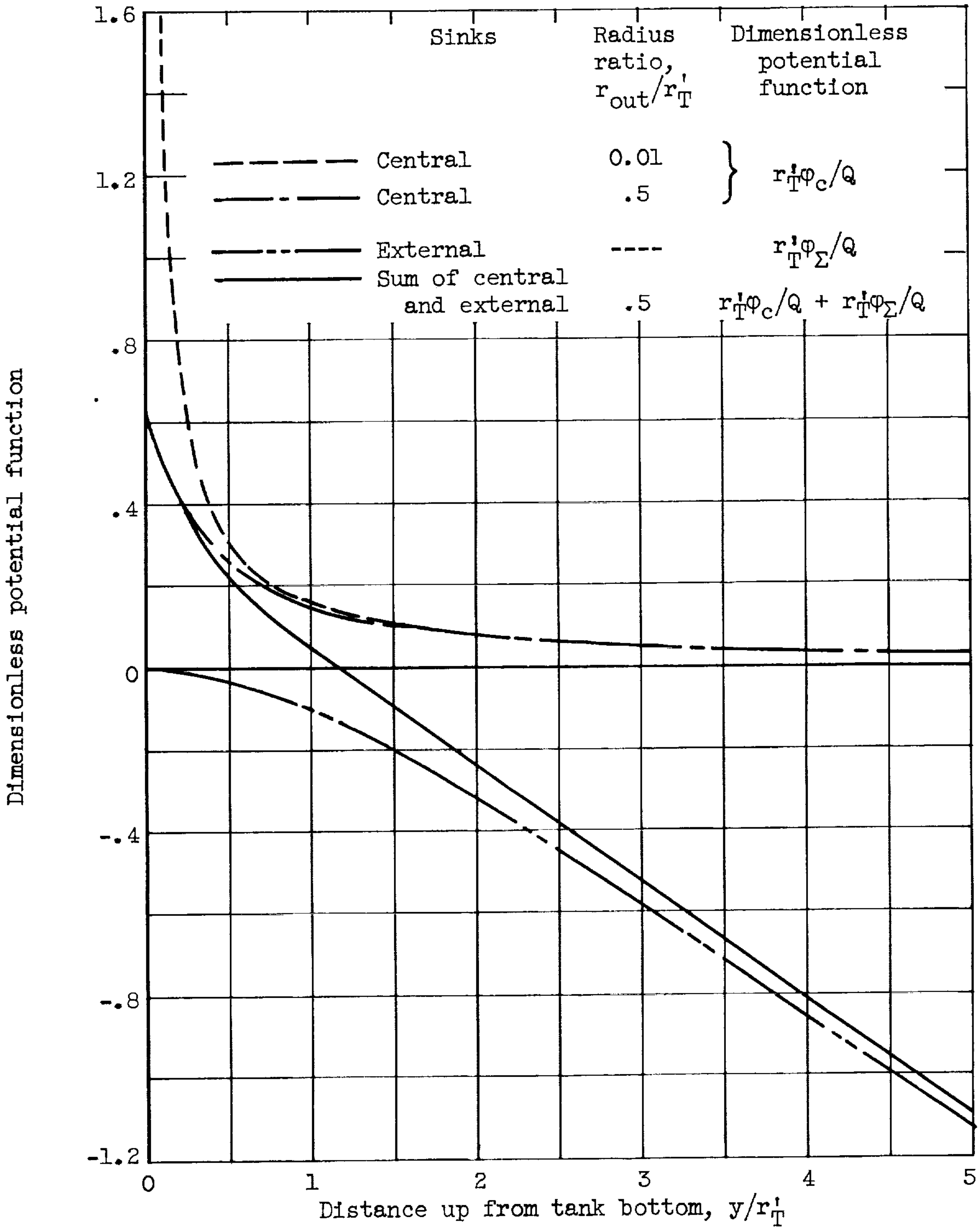
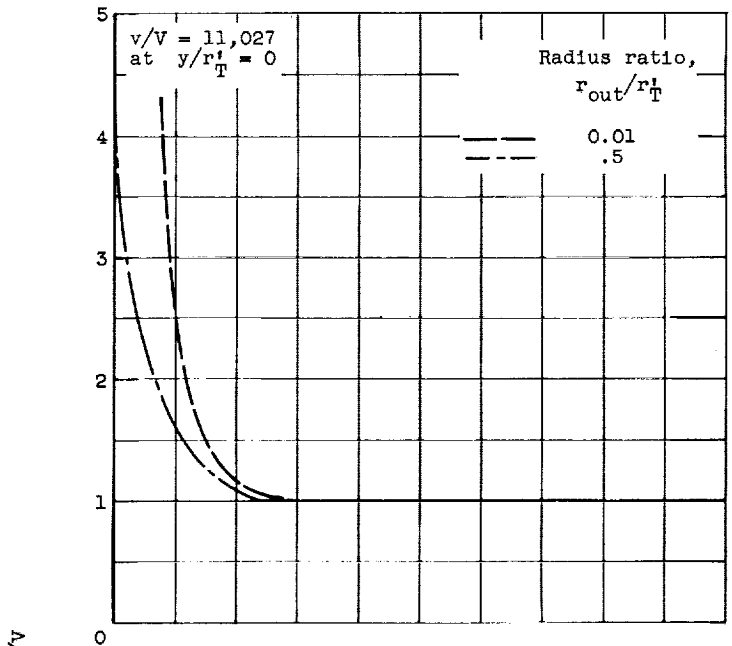
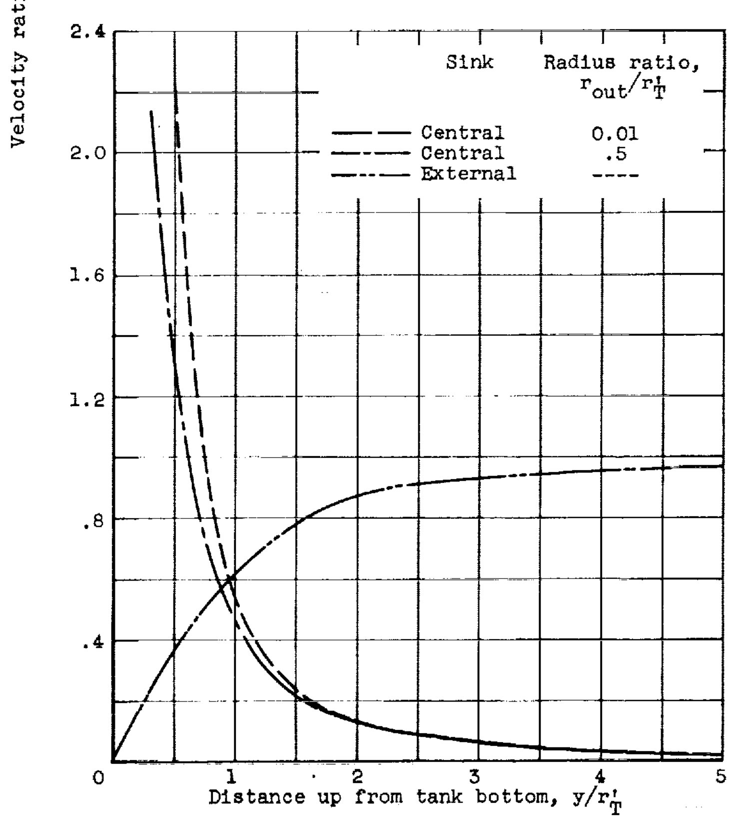


Figure 3. - Contribution of central distributed sink and external point sinks to potential function.



(a) Sum of velocities induced by central sink and external sinks.



(b) Separate contributions of central and external sinks to induced velocity.

Figure 4. - Velocity distribution along tank centerline.

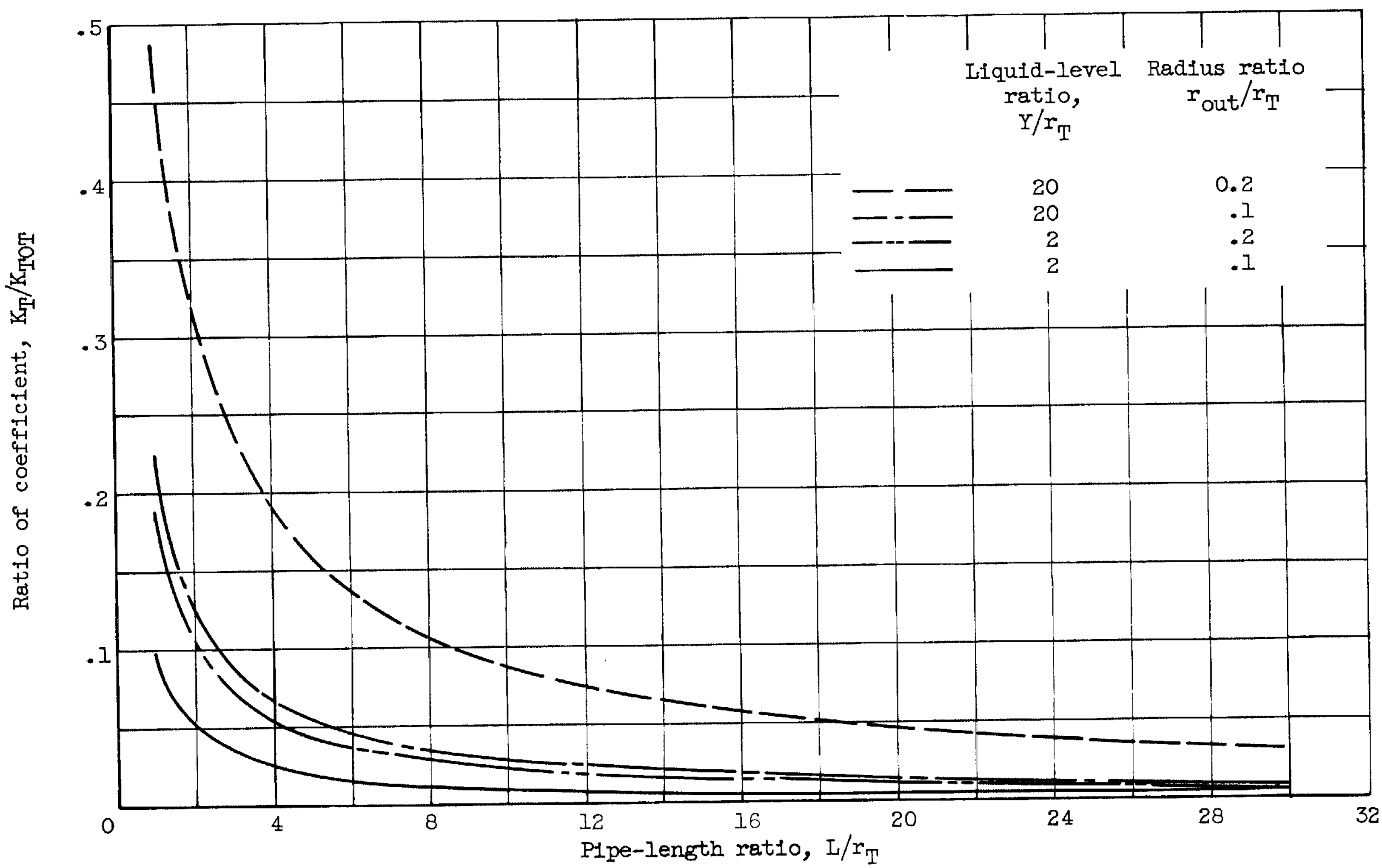


Figure 5. - **Variation of coefficient ratio indicative of significance of tank dynamics in system** with outlet pipe of length L and radius r_{out}. Liquid level, Y; tank radius, r_T.

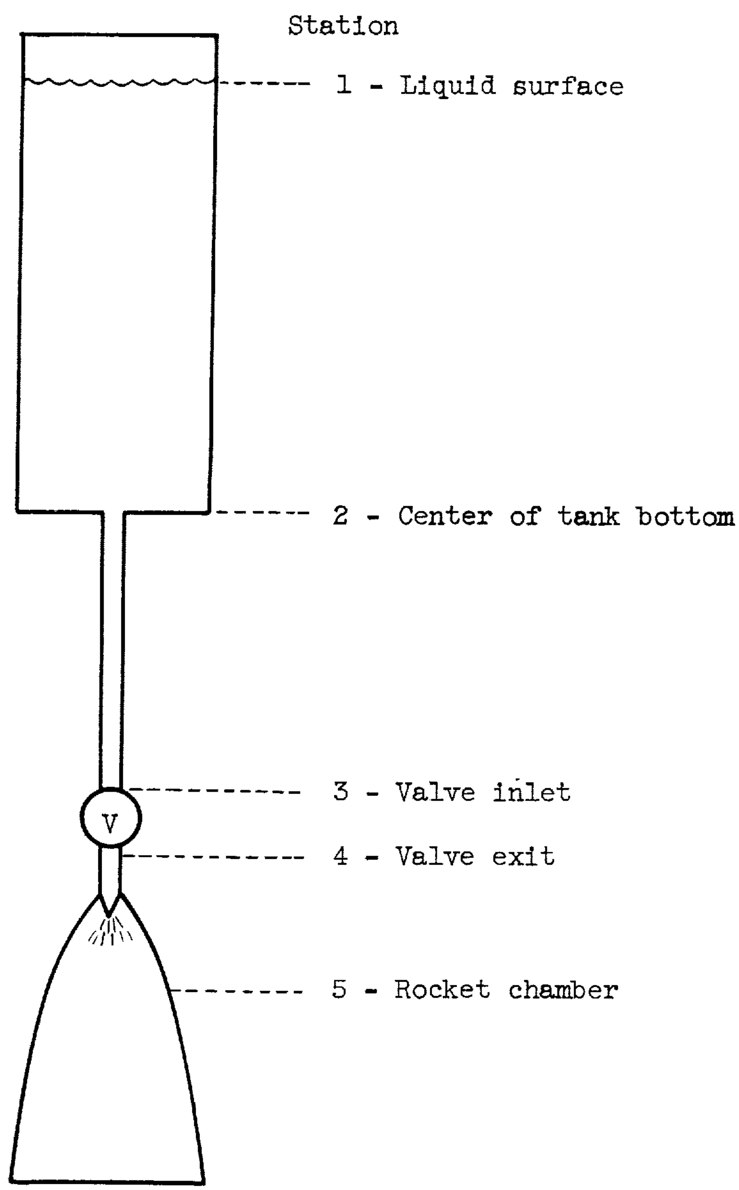


Figure 6. - Schematic diagram of part of simple rocket system.

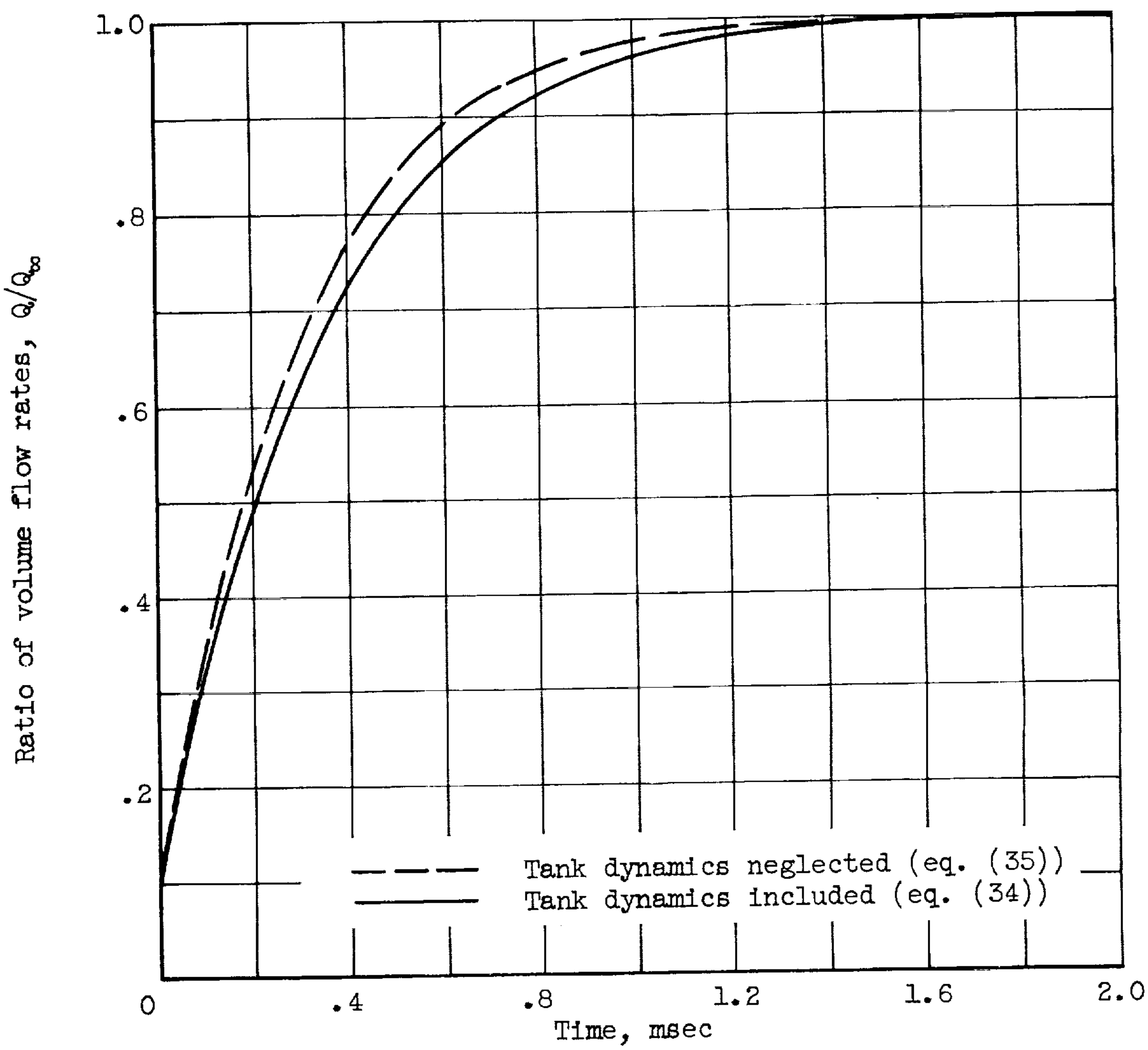


Figure 7. - Comparison of development of volume flow rate including and neglecting unsteady tank flow.

NASA TN D-1164
National Aeronautics and Space Administration.
POTENTIAL-FLOW ANALYSIS OF UNSTEADY OUTFLOW FROM A TANK AND ITS EFFECT ON THE DYNAMICS OF A FLUID SYSTEM. William H. Roudebush and I. Irving Pinkel. June 1962. 25p. OTS price, $0.75.
(NASA TECHNICAL NOTE D-1164)

When an analytical investigation of the unsteady flow from a propellant tank was made, a simple equation was obtained that relates the tank pressure drop to the development of the volume flow rate out of the tank. This equation allows the incorporation of the tank characteristics into a one-dimensional analysis of the dynamics of a fluid system and makes possible the determination of situations in which the dynamic characteristics of the tank are important.

I. Roudebush, William H.
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